8.

Scalar product

- Scalar product
- Algebraic properties of the scalar product
- The scalar product from the components
- Proofs using the scalar product
- Miscellaneous exercise eight

So far in our consideration of vectors we have not developed any way of testing whether two vectors, $a\mathbf{i} + b\mathbf{j}$ and $c\mathbf{i} + d\mathbf{j}$, are perpendicular.

This chapter considers a concept called the **scalar product** of two vectors and this does give such a test for perpendicularity between two vectors.

The idea of forming a product of two vectors may seem rather confusing initially. How do we multiply together quantities which have magnitude and direction? Well we could define what we mean by vector multiplication in all sorts of ways but there are two methods of performing vector multiplication that prove to be useful. One method of vector multiplication gives an answer that is a scalar. We call this the **scalar product** and will consider the definition of this product in a moment. A second method gives an answer that is a vector. We call this the **vector product**, a concept that is beyond the scope of this unit.

Scalar product

Component form of the dot product

Parallel and perpendicular vectors

We define the scalar product of two vectors **a** and **b** to be the magnitude of **a** multiplied by the magnitude of **b** multiplied by the cosine of the angle between **a** and **b**. We write this product as **a.b** and say this as 'a dot b'. For this reason the scalar product is also referred to as the 'dot product'.

Thus

$$\mathbf{a}.\mathbf{b} = |\mathbf{a}|\ |\mathbf{b}| \cos\theta \qquad \text{where } \theta \text{ is the angle between } \mathbf{a} \text{ and } \mathbf{b}.$$

Note

- It follows from the above definition that for two perpendicular vectors,

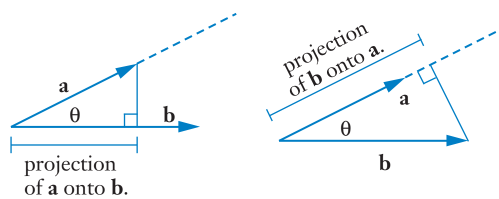
$$\mathbf{a}.\mathbf{b} = 0.$$

- '$|\mathbf{a}| \cos\theta$' is the **scalar projection** of **a** onto **b** or the **resolved part** of **a** in the direction of **b**.

  The **vector projection** of **a** onto **b** is

$$|\mathbf{a}| \cos\theta\, \hat{\mathbf{b}}$$

  (Remember, $\hat{\mathbf{b}}$ is a unit vector in the direction of **b**.)



  Similarly '$|\mathbf{b}| \cos\theta$' can be referred to as the scalar projection of **b** onto **a** and the vector projection of **b** onto **a** is $|\mathbf{b}| \cos\theta\, \hat{\mathbf{a}}$.

- Remember that the angle between two vectors refers to the angle between the vectors when they are either both directed away from a point or both directed towards it.

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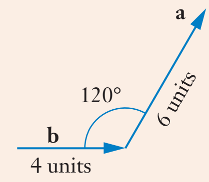
EXAMPLE 1

With **a** and **b** as defined in the diagram determine **a.b**.



Solution

$$\begin{aligned}\mathbf{a.b} &= |\mathbf{a}|\ |\mathbf{b}|\cos\theta\\ &= (6)\ (4)\cos 60°\\ &= 12\end{aligned}$$

EXAMPLE 2

Given that $|\mathbf{a}| = 7$, $|\mathbf{b}| = 2$ and $\mathbf{a.b} = 10$, find the angle between **a** and **b**, correct to the nearest degree.

Solution

$$\mathbf{a.b} = |\mathbf{a}|\ |\mathbf{b}|\cos\theta$$

$$\therefore\quad 10 = (7)\ (2)\cos\theta$$

$$\cos\theta \approx 0.7143$$

$$\therefore\quad \theta \approx 44°$$

The angle between **a** and **b** is 44°, correct to the nearest degree.

Properties of the dot product

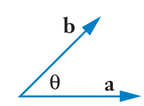
Algebraic properties of the scalar product

A number of algebraic properties of the scalar product follow as a consequence of the way we define the scalar product, **a.b**.

It follows that: $\mathbf{a.b} = \mathbf{b.a}$

Proof: $$\begin{aligned}\mathbf{a.b} &= |\mathbf{a}|\ |\mathbf{b}|\cos\theta\\ &= |\mathbf{b}|\ |\mathbf{a}|\cos\theta\\ &= \mathbf{b.a}\end{aligned}$$

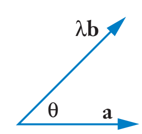
It also follows that: $$\begin{aligned}\mathbf{a}.(\lambda\mathbf{b}) &= \lambda(\mathbf{a.b})\\ &= (\lambda\mathbf{a}).\mathbf{b}\end{aligned}$$



Proof: If $\lambda > 0$

$$\begin{aligned}\mathbf{a}.(\lambda\mathbf{b}) &= |\mathbf{a}|\ |\lambda\mathbf{b}|\cos\theta\\ &= |\mathbf{a}|\ |\lambda|\ |\mathbf{b}|\cos\theta\\ &= \lambda\ |\mathbf{a}|\ |\mathbf{b}|\cos\theta\\ &= \lambda(\mathbf{a.b})\end{aligned}$$

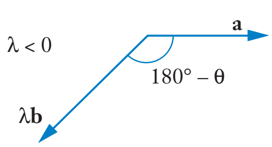
$\lambda > 0$



If $\lambda < 0$

$$\begin{aligned}\mathbf{a}.(\lambda\mathbf{b}) &= |\mathbf{a}|\ |\lambda\mathbf{b}|\cos(180° - \theta)\\ &= -|\mathbf{a}|\ |\lambda|\ |\mathbf{b}|\cos\theta\\ &= -|\lambda|\ |\mathbf{a}|\ |\mathbf{b}|\cos\theta\\ &= \lambda\ |\mathbf{a}|\ |\mathbf{b}|\cos\theta\\ &= \lambda(\mathbf{a.b})\end{aligned}$$

$\lambda < 0$



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Another property is: $\mathbf{a.a} = |\mathbf{a}|^2$

If we write $|\mathbf{a}|$ simply as a this last result can be written as follows:

$$\mathbf{a.a} = a^2$$

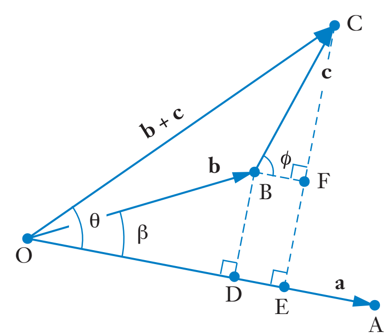
Proof:

$$\begin{aligned}\mathbf{a.a} &= |\mathbf{a}|\ |\mathbf{a}| \cos 0° \\ &= |\mathbf{a}|\ |\mathbf{a}|\ (1) \\ &= |\mathbf{a}|^2 \\ &= a^2\end{aligned}$$

An important result is: $\mathbf{a}.(\mathbf{b} + \mathbf{c}) = \mathbf{a.b} + \mathbf{a.c}$

Proof: The diagram on the right shows $\mathbf{a}$ ($= \overrightarrow{OA}$), $\mathbf{b}$ ($= \overrightarrow{OB}$) and $\mathbf{c}$ ($= \overrightarrow{BC}$). Thus $\overrightarrow{OC} = \mathbf{b} + \mathbf{c}$.

The angle between $\mathbf{a}$ and $\mathbf{b}$ is β, between $\mathbf{a}$ and $\mathbf{c}$ is ϕ and between $\mathbf{a}$ and $(\mathbf{b} + \mathbf{c})$ is θ.



$$\begin{aligned}\mathbf{a}.(\mathbf{b} + \mathbf{c}) &= |\mathbf{a}|\ |\mathbf{b} + \mathbf{c}| \cos\theta \\ &= |\mathbf{a}|\ (OC) \cos\theta \\ &= |\mathbf{a}|\ (OE) \\ &= |\mathbf{a}|\ (OD + DE) \\ &= |\mathbf{a}|\ (OD) + |\mathbf{a}|\ (DE)\end{aligned}$$

But $OD = |\mathbf{b}| \cos\beta$ and $DE = BF = |\mathbf{c}| \cos\phi$

$\therefore$ $$\begin{aligned}\mathbf{a}.(\mathbf{b} + \mathbf{c}) &= |\mathbf{a}|\ |\mathbf{b}| \cos\beta + |\mathbf{a}|\ |\mathbf{c}| \cos\phi \\ &= \mathbf{a.b} + \mathbf{a.c}\end{aligned}$$

Similarly: $(\mathbf{a} + \mathbf{b}).\mathbf{c} = \mathbf{a.c} + \mathbf{b.c}$

and so: $(\mathbf{a} + \mathbf{b}).(\mathbf{c} + \mathbf{d}) = \mathbf{a.c} + \mathbf{a.d} + \mathbf{b.c} + \mathbf{b.d}$

Summary:

$$\mathbf{a.b} = \mathbf{b.a}$$
$$\mathbf{a}.(\lambda\mathbf{b}) = (\lambda\mathbf{a}).\mathbf{b} = \lambda(\mathbf{a.b})$$
$$\mathbf{a.a} = a^2$$
$$\mathbf{a}.(\mathbf{b} + \mathbf{c}) = \mathbf{a.b} + \mathbf{a.c}$$
$$(\mathbf{a} + \mathbf{b}).\mathbf{c} = \mathbf{a.c} + \mathbf{b.c}$$
$$(\mathbf{a} + \mathbf{b}).(\mathbf{c} + \mathbf{d}) = \mathbf{a.c} + \mathbf{a.d} + \mathbf{b.c} + \mathbf{b.d}$$

EXAMPLE 3

a Expand and simplify $(2\mathbf{p} + \mathbf{q}).(5\mathbf{p} - 2\mathbf{q})$.

b Simplify $(\mathbf{u} + 2\mathbf{v}).(\mathbf{u} - \mathbf{v})$ given that $\mathbf{u}$ is perpendicular to $\mathbf{v}$.

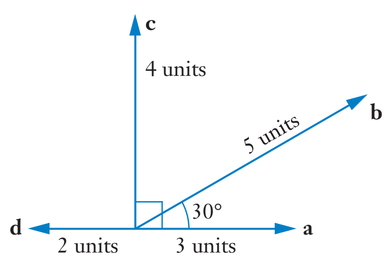
Solution

a
$$\begin{aligned}(2\mathbf{p} + \mathbf{q}).(5\mathbf{p} - 2\mathbf{q}) &= 2\mathbf{p}.5\mathbf{p} + 2\mathbf{p}.(-2\mathbf{q}) + \mathbf{q}.5\mathbf{p} + \mathbf{q}.(-2\mathbf{q})\\ &= 10\,\mathbf{p}.\mathbf{p} - 4\,\mathbf{p}.\mathbf{q} + 5\,\mathbf{q}.\mathbf{p} - 2\,\mathbf{q}.\mathbf{q}\\ &= 10p^2 - 4\mathbf{p}.\mathbf{q} + 5\mathbf{p}.\mathbf{q} - 2q^2\\ &= 10p^2 + \mathbf{p}.\mathbf{q} - 2q^2\end{aligned}$$

b
$$\begin{aligned}(\mathbf{u} + 2\mathbf{v}).(\mathbf{u} - \mathbf{v}) &= \mathbf{u}.\mathbf{u} - \mathbf{u}.\mathbf{v} + 2\mathbf{v}.\mathbf{u} - 2\mathbf{v}.\mathbf{v}\\ &= u^2 + \mathbf{u}.\mathbf{v} - 2v^2\\ &= u^2 - 2v^2 \text{ (because } \mathbf{u} \text{ is perpendicular to } \mathbf{v} \text{ and hence } \mathbf{u}.\mathbf{v} = 0).\end{aligned}$$

Exercise 8A

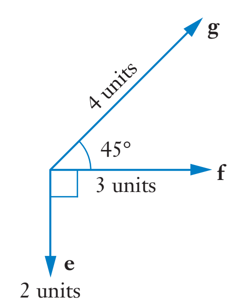
In questions **1** to **6** evaluate the given scalar product where **a**, **b**, **c** and **d** are as shown in the diagram. (Leave answers in exact form.)



1 **a.b** **2** **a.c** **3** **a.d**

4 **b.c** **5** **b.d** **6** **c.d**

In questions **7** to **12** evaluate the given scalar product where **e**, **f** and **g** are as shown in the diagram. (Leave answers in exact form.)



7 **e.f** **8** **f.e** **9** **e.e**

10 **f.f** **11** **f.g** **12** **g.f**

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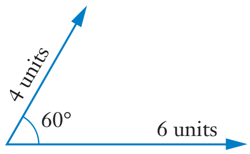
In questions **13** to **18** find the scalar product of the two vectors shown. (Leave answers in exact form).

13



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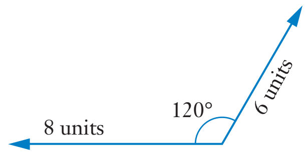

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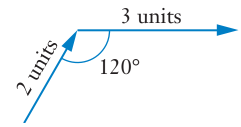

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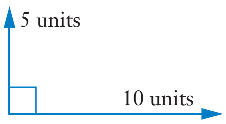

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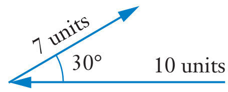

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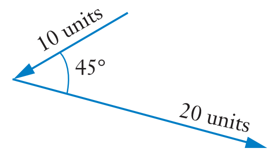

19 For each of the following, state whether scalar or vector.

a $|\mathbf{a}|$ **b** $\mathbf{a}.\mathbf{b}$ **c** $\mathbf{a} + \mathbf{b}$ **d** $\mathbf{a} - \mathbf{b}$

e $\mathbf{a} + 2\mathbf{b}$ **f** $\mathbf{a}.(2\mathbf{b})$ **g** $(\mathbf{a} + \mathbf{b}).(\mathbf{c} + \mathbf{d})$ **h** $|\mathbf{a} + \mathbf{b}|$

i $\mathbf{a} + \lambda\mathbf{b}$ **j** $\mathbf{a}.(\lambda\mathbf{b})$

20 With **i** and **j** representing horizontal and vertical unit vectors respectively, find

a $\mathbf{i}.\mathbf{i}$ **b** $\mathbf{i}.\mathbf{j}$ **c** $\mathbf{j}.\mathbf{j}$

21 Expand and simplify each of the following. (Writing $|\mathbf{a}|$ as a and $|\mathbf{b}|$ as b.)

a $(\mathbf{a} + \mathbf{b}).(\mathbf{a} - \mathbf{b})$ **b** $(\mathbf{a} + \mathbf{b}).(\mathbf{a} + \mathbf{b})$ **c** $(\mathbf{a} - \mathbf{b}).(\mathbf{a} - \mathbf{b})$

d $(2\mathbf{a} + \mathbf{b}).(2\mathbf{a} - \mathbf{b})$ **e** $(\mathbf{a} + 3\mathbf{b}).(\mathbf{a} - 2\mathbf{b})$ **f** $\mathbf{a}.(\mathbf{a} - \mathbf{b}) + \mathbf{a}.\mathbf{b}$

22 If **a** and **b** are perpendicular vectors prove that $(\mathbf{a} + \mathbf{b}).(\mathbf{a} - 2\mathbf{b}) = a^2 - 2b^2$, where a is $|\mathbf{a}|$ and b is $|\mathbf{b}|$.

23 If **a** and **b** are perpendicular vectors, and writing $|\mathbf{a}|$ as a and $|\mathbf{b}|$ as b, which of the following statements are necessarily true?

a $a = b$ **b** $\mathbf{a}.\mathbf{b} = 0$ **c** $ab = 0$ **d** $\mathbf{a}.(\mathbf{a} + \mathbf{b}) = a^2$

24 If **a** is perpendicular to $(\mathbf{b} - \mathbf{c})$ which of the following statements are necessarily true?

a $\mathbf{a}.(\mathbf{b} - \mathbf{c}) = 0$ **b** $\mathbf{a}.\mathbf{b} = \mathbf{a}.\mathbf{c}$

c $\mathbf{b} = \mathbf{c}$ **d** **a** is perpendicular to $(\mathbf{c} - \mathbf{b})$

25 Find an expression for $\mathbf{a}.\mathbf{b}$ given that $\mathbf{a} = x_1\mathbf{i} + y_1\mathbf{j}$, $\mathbf{b} = x_2\mathbf{i} + y_2\mathbf{j}$ and **i** and **j** are perpendicular vectors each of unit length.

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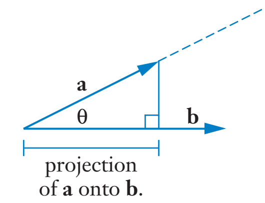
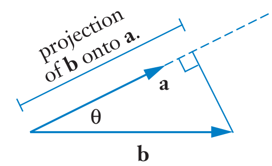
26 Find the scalar projection of **a** onto **b** given that

$$\mathbf{a}.\mathbf{b} = 14 \text{ and } |\mathbf{b}| = 5.$$



27 Find the scalar projection of **b** onto **a** given that

$$\mathbf{a}.\mathbf{b} = 18 \text{ and } |\mathbf{a}| = 25.$$



28 If **a** and **b** are perpendicular vectors which of the following statements are necessarily true?

a $\mathbf{a}.(\mathbf{a} - \mathbf{b}) = 0$ **b** $(\mathbf{a} + \mathbf{b}).(\mathbf{a} - \mathbf{b}) = 0$

c $(\mathbf{a} + \mathbf{b}).(\mathbf{a} + \mathbf{b}) = \mathrm{a}^2 + \mathrm{b}^2$

29 Given that $|\mathbf{a}| = 5$, $|\mathbf{b}| = 3$ and $\mathbf{a}.\mathbf{b} = 7$, find:

a the angle between **a** and **b**, correct to the nearest degree,

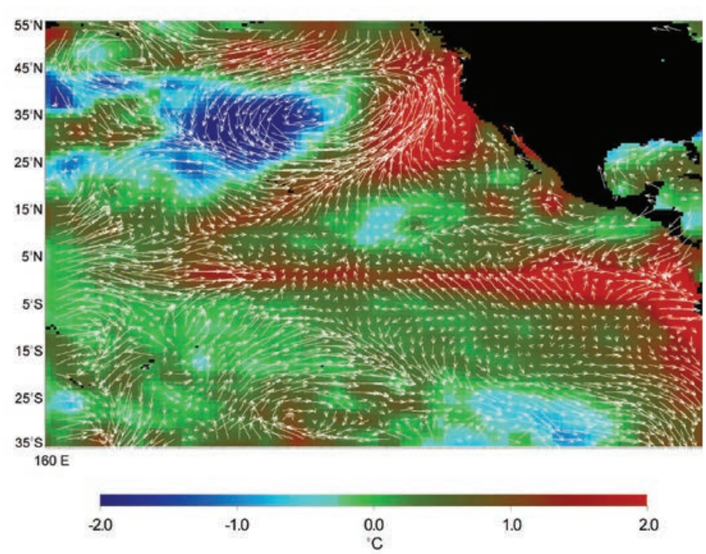
b $\mathbf{a}.\mathbf{a}$ **c** $\mathbf{b}.\mathbf{b}$ **d** $(\mathbf{a} - \mathbf{b}).(\mathbf{a} - \mathbf{b})$ **e** $|\mathbf{a} - \mathbf{b}|$

30 Evaluate $\mathbf{p}.\mathbf{q}$ given that $\mathbf{p} = 3\mathbf{a} + 2\mathbf{b}$, $\mathbf{q} = 4\mathbf{a} - \mathbf{b}$, $|\mathbf{a}| = 3$, $|\mathbf{b}| = 2$ and **a** and **b** are perpendicular to each other.

31 Explain why with three vectors **a**, **b** and **c**, we can talk of $\mathbf{a}.(\mathbf{b} + \mathbf{c})$ and also of $\mathbf{a}.(\mathbf{b} - \mathbf{c})$ but $\mathbf{a}.(\mathbf{b}.\mathbf{c})$ has no meaning.

32 **a** Prove that $|\mathbf{a}.\mathbf{b}| \le |\mathbf{a}|\ |\mathbf{b}|$

b Expand $(\mathbf{a} + \mathbf{b}).(\mathbf{a} + \mathbf{b})$ and use this, together with the inequality from **a** to prove the triangle inequality: $|\mathbf{a} + \mathbf{b}| \le |\mathbf{a}| + |\mathbf{b}|$



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The scalar product from the components

If you managed question **25** of the previous exercise then you would have obtained an important result that allows $\mathbf{a}.\mathbf{b}$ to be determined quickly when $\mathbf{a}$ and $\mathbf{b}$ are given in component form. Your working for question **25** should have followed the same reasoning as is shown below.

Let $\mathbf{a} = x_1\mathbf{i} + y_1\mathbf{j}$ and $\mathbf{b} = x_2\mathbf{i} + y_2\mathbf{j}$.

Then
$$\begin{aligned}\mathbf{a}.\mathbf{b} &= (x_1\mathbf{i} + y_1\mathbf{j}).(x_2\mathbf{i} + y_2\mathbf{j})\\ &= (x_1\mathbf{i}).(x_2\mathbf{i}) + (x_1\mathbf{i}).(y_2\mathbf{j}) + (y_1\mathbf{j}).(x_2\mathbf{i}) + (y_1\mathbf{j}).(y_2\mathbf{j})\\ &= x_1x_2(\mathbf{i}.\mathbf{i}) + x_1y_2(\mathbf{i}.\mathbf{j}) + y_1x_2(\mathbf{j}.\mathbf{i}) + y_1y_2(\mathbf{j}.\mathbf{j}) \qquad [1]\end{aligned}$$

But
$$\begin{aligned}\mathbf{i}.\mathbf{i} &= (1)(1)\cos 0° \\ &= 1\end{aligned} \qquad \begin{aligned}\mathbf{i}.\mathbf{j} &= \mathbf{j}.\mathbf{i}\\ &= (1)(1)\cos 90°\\ &= 0\end{aligned} \qquad \begin{aligned}\mathbf{j}.\mathbf{j} &= (1)(1)\cos 0°\\ &= 1\end{aligned}$$

$\therefore$ from [1] $\mathbf{a}.\mathbf{b} = x_1x_2 + y_1y_2$

> If $\mathbf{a} = x_1\mathbf{i} + y_1\mathbf{j}$ and $\mathbf{b} = x_2\mathbf{i} + y_2\mathbf{j}$ then $\mathbf{a}.\mathbf{b} = x_1x_2 + y_1y_2$
> and, by definition, $\mathbf{a}.\mathbf{b} = |\mathbf{a}|\ |\mathbf{b}| \cos\theta$

EXAMPLE 4

Given that $\mathbf{u} = 5\mathbf{i} - 3\mathbf{j}$, $\mathbf{v} = 2\mathbf{i} + \mathbf{j}$ and $\mathbf{w} = -4\mathbf{i} + 3\mathbf{j}$, find

a $\mathbf{u}.\mathbf{v}$ **b** $\mathbf{u}.\mathbf{w}$ **c** $\mathbf{u}.(2\mathbf{v} - \mathbf{w})$

Solution

a
$$\begin{aligned}\mathbf{u}.\mathbf{v} &= (5\mathbf{i} - 3\mathbf{j}).(2\mathbf{i} + \mathbf{j})\\ &= (5)(2) + (-3)(1)\\ &= 7\end{aligned}$$

b
$$\begin{aligned}\mathbf{u}.\mathbf{w} &= (5\mathbf{i} - 3\mathbf{j}).(-4\mathbf{i} + 3\mathbf{j})\\ &= (5)(-4) + (-3)(3)\\ &= -29\end{aligned}$$

c
$$\begin{aligned}2\mathbf{v} - \mathbf{w} &= 2(2\mathbf{i} + \mathbf{j}) - (-4\mathbf{i} + 3\mathbf{j})\\ &= 8\mathbf{i} - \mathbf{j}\end{aligned}$$

$$\begin{aligned}\therefore\quad \mathbf{u}.(2\mathbf{v} - \mathbf{w}) &= (5\mathbf{i} - 3\mathbf{j}).(8\mathbf{i} - \mathbf{j})\\ &= (5)(8) + (-3)(-1)\\ &= 43\end{aligned}$$

Or, using parts **a** and **b**:
$$\begin{aligned}\mathbf{u}.(2\mathbf{v} - \mathbf{w}) &= 2\mathbf{u}.\mathbf{v} - \mathbf{u}.\mathbf{w}\\ &= 2 \times 7 - (-29)\\ &= 14 + 29\\ &= 43, \text{ as before.}\end{aligned}$$

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EXAMPLE 5

Find the value of λ given that **a** (= 2**i** + 3**j**) is perpendicular to **b** (= λ**i** – 5**j**).

Solution

a and **b** are perpendicular. Thus $\quad \mathbf{a}.\mathbf{b} = 0$

$\therefore \qquad (2\mathbf{i} + 3\mathbf{j}).(\lambda\mathbf{i} - 5\mathbf{j}) = 0$

i.e. $\qquad 2\lambda - 15 = 0$

giving $\qquad \lambda = 7.5$

With **a** and **b** perpendicular λ must equal 7.5.

EXAMPLE 6

Find the angle between **a** and **b** given that **a** = 5**i** + 2**j** and **b** = –3**i** + **j**. (Give your answer correct to the nearest degree.)

Solution

$$\begin{aligned}\mathbf{a}.\mathbf{b} &= (5\mathbf{i} + 2\mathbf{j}).(-3\mathbf{i} + \mathbf{j})\\ &= (5)(-3) + (2)(1)\\ &= -13\end{aligned}$$

$$\begin{aligned}|\mathbf{a}| &= \sqrt{5^2 + 2^2}\\ &= \sqrt{29}\end{aligned}$$

$$\begin{aligned}|\mathbf{b}| &= \sqrt{(-3)^2 + 1^2}\\ &= \sqrt{10}\end{aligned}$$

But $\qquad \mathbf{a}.\mathbf{b} = |\mathbf{a}|\ |\mathbf{b}| \cos\theta$

$\therefore \qquad -13 = \sqrt{29}\sqrt{10} \cos\theta$

giving $\qquad \theta \approx 139.8°$.

The angle between **a** and **b** is 140°, correct to the nearest degree.

TECHNOLOGY

Some calculators and computer programs are able to determine the scalar product of two vectors directly, as well as give other information about vectors, as suggested by the display.

$$\text{dotP}\left(\begin{bmatrix}5\\2\end{bmatrix}, \begin{bmatrix}-3\\1\end{bmatrix}\right)$$

$$-13$$

$$\text{angle}\left(\begin{bmatrix}5\\2\end{bmatrix}, \begin{bmatrix}-3\\1\end{bmatrix}\right)$$

$$139.7636417$$

$$\text{solve}\left(\text{dotP}\left(\begin{bmatrix}2\\3\end{bmatrix}, \begin{bmatrix}\lambda\\-5\end{bmatrix}\right) = 0, \lambda\right)$$

$$\lambda = 7.5$$

Explore your calculator in this regard.

However, make sure that you can show how to obtain the scalar product and the angle between vectors algebraically if required to do so.

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Exercise 8B

1 Given that $\mathbf{a} = 3\mathbf{i} - 2\mathbf{j}$, $\mathbf{b} = 5\mathbf{i} + 6\mathbf{j}$ and $\mathbf{c} = 2\mathbf{i} - \mathbf{j}$ evaluate the following.

a $\mathbf{a}.\mathbf{b}$ **b** $\mathbf{b}.\mathbf{a}$ **c** $\mathbf{a}.\mathbf{c}$ **d** $\mathbf{b}.\mathbf{c}$

2 Given that $\mathbf{x} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$, $\mathbf{y} = \begin{pmatrix} 5 \\ -1 \end{pmatrix}$ and $\mathbf{z} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$ evaluate the following.

a $\mathbf{x}.\mathbf{y}$ **b** $\mathbf{x}.\mathbf{z}$ **c** $\mathbf{z}.\mathbf{x}$ **d** $\mathbf{y}.\mathbf{z}$

3 Given that $\mathbf{p} = <3, 1>$, $\mathbf{q} = <2, -1>$ and $\mathbf{r} = <5, 2>$ evaluate the following.

a $\mathbf{q}.\mathbf{r}$ **b** $2\mathbf{q}.3\mathbf{r}$ **c** $\mathbf{p}.(\mathbf{q} + \mathbf{r})$ **d** $\mathbf{p}.(\mathbf{q} - \mathbf{r})$

4 State whether each pair of vectors are perpendicular vectors or not.

a $2\mathbf{i} + 3\mathbf{j}$ and $4\mathbf{i} - 2\mathbf{j}$ **b** $-2\mathbf{i} + \mathbf{j}$ and $4\mathbf{i} - 2\mathbf{j}$

c $3\mathbf{i} - \mathbf{j}$ and $2\mathbf{i} + 6\mathbf{j}$ **d** $<12, -3>$ and $<1, 4>$

e $\begin{pmatrix} 5 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} -3 \\ 7 \end{pmatrix}$ **f** $\begin{pmatrix} 14 \\ 8 \end{pmatrix}$ and $\begin{pmatrix} -4 \\ 7 \end{pmatrix}$

5 Given that $\mathbf{d} = 3\mathbf{i} + \mathbf{j}$, $\mathbf{e} = 2\mathbf{i} + 4\mathbf{j}$ and $\mathbf{f} = -2\mathbf{i} - 3\mathbf{j}$, evaluate the following.

a $\mathbf{d}.\mathbf{e}$ **b** $\mathbf{e}.\mathbf{f}$ **c** $\mathbf{d}.(\mathbf{e} + \mathbf{f})$ **d** $(\mathbf{d} + \mathbf{e}).\mathbf{f}$

6 Given that $\mathbf{a} = 2\mathbf{i} - \mathbf{j}$, $\mathbf{b} = 3\mathbf{i} + 2\mathbf{j}$ and $\mathbf{c} = 4\mathbf{i} - 3\mathbf{j}$, evaluate the following.

a $\mathbf{a}.(\mathbf{b} + \mathbf{c})$ **b** $(\mathbf{a} + \mathbf{b}).\mathbf{c}$ **c** $\mathbf{b}.(\mathbf{a} + \mathbf{c})$ **d** $(\mathbf{a} - \mathbf{b}).(\mathbf{b} - \mathbf{c})$

7 If $\mathbf{a} = 2\mathbf{i} + 3\mathbf{j}$, $\mathbf{b} = \mathbf{i} - 4\mathbf{j}$ and $\mathbf{c} = -4\mathbf{i} + 5\mathbf{j}$ evaluate

a $\mathbf{a}.\mathbf{b}$ **b** $\mathbf{a}.\mathbf{c}$ **c** $\mathbf{b} + \mathbf{c}$

Use your answer to part **c** to determine $\mathbf{a}.(\mathbf{b} + \mathbf{c})$ and hence verify that

$$\mathbf{a}.(\mathbf{b} + \mathbf{c}) = \mathbf{a}.\mathbf{b} + \mathbf{a}.\mathbf{c}$$

8 If $\mathbf{p} = 3\mathbf{i} + 4\mathbf{j}$ and $\mathbf{q} = 5\mathbf{i} - 12\mathbf{j}$ find

a $|\mathbf{p}|$ **b** $|\mathbf{q}|$ **c** $\mathbf{p}.\mathbf{q}$

Hence find the angle between **p** and **q**, to the nearest degree.

9 If $\mathbf{c} = 7\mathbf{i} + 7\mathbf{j}$ and $\mathbf{d} = 15\mathbf{i} - 8\mathbf{j}$ find

a $|\mathbf{c}|$ **b** $|\mathbf{d}|$ **c** $\mathbf{c}.\mathbf{d}$

Hence find the angle between **c** and **d**, to the nearest degree.

10 If $\mathbf{a} = 2\mathbf{i} - 3\mathbf{j}$, $\mathbf{b} = 4\mathbf{i} + 5\mathbf{j}$ and $\mathbf{c} = 2\mathbf{i} - \mathbf{j}$, prove that **a** is perpendicular to $(\mathbf{b} + \mathbf{c})$.

11 If $\mathbf{a} = -2\mathbf{i} + 2\mathbf{j}$, $\mathbf{b} = 5\mathbf{i} + 2\mathbf{j}$ and $\mathbf{c} = 4\mathbf{i} - \mathbf{j}$, prove that $(\mathbf{a} + 2\mathbf{b})$ and $(\mathbf{b} - 2\mathbf{c})$ are perpendicular vectors.

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For each of numbers **12** to **17** find:

a the scalar product of the two vectors,

b the angle between the two vectors (to the nearest degree).

12 $\mathbf{a} = 3\mathbf{i} + 4\mathbf{j}$, $\mathbf{b} = 4\mathbf{i} + 3\mathbf{j}$.

13 $\mathbf{c} = 24\mathbf{i} + 7\mathbf{j}$, $\mathbf{d} = 5\mathbf{i} + 12\mathbf{j}$.

14 $\mathbf{e} = 2\mathbf{i} + \mathbf{j}$, $\mathbf{f} = 3\mathbf{i} - 2\mathbf{j}$.

15 $\mathbf{g} = 2\mathbf{i} + \mathbf{j}$, $\mathbf{h} = -4\mathbf{i} + 8\mathbf{j}$.

16 $\mathbf{m} = -3\mathbf{i} + 4\mathbf{j}$, $\mathbf{n} = 9\mathbf{i} - 12\mathbf{j}$.

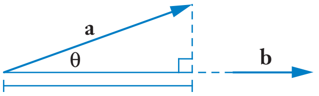
17 $\mathbf{p} = \mathbf{i}$, $\mathbf{q} = 12\mathbf{i} - 5\mathbf{j}$.

18 $\mathbf{a} = 2\mathbf{i} + 3\mathbf{j}$, $\mathbf{b} = \lambda\mathbf{i} + 12\mathbf{j}$ and $\mathbf{c} = \mu\mathbf{i} - 7\mathbf{j}$.
Find λ given that $\mathbf{a}$ and $\mathbf{b}$ are parallel.
Find μ given that $\mathbf{a}$ and $\mathbf{c}$ are perpendicular.

19 $\mathbf{d} = w\mathbf{i} + \mathbf{j}$, $\mathbf{e} = -\mathbf{i} + 7\mathbf{j}$ and $\mathbf{f} = x\mathbf{i} + 5\mathbf{j}$.
Find w given that $\mathbf{d}$ and $\mathbf{e}$ are perpendicular.
Find x given that it is a negative number and that $|\mathbf{d}| = |\mathbf{f}|$.

20 The scalar projection of $\mathbf{a}$ onto $\mathbf{b}$ is $|\mathbf{a}| \cos\theta$,

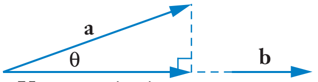
i.e. $\dfrac{\mathbf{a}.\mathbf{b}}{|\mathbf{b}|}$



Scalar projection of **a** onto **b**.

The vector projection of $\mathbf{a}$ onto $\mathbf{b}$ is $|\mathbf{a}| \cos\theta\, \hat{\mathbf{b}}$,

i.e. $\dfrac{\mathbf{a}.\mathbf{b}}{\mathbf{b}.\mathbf{b}}\,\mathbf{b}$



Vector projection of **a** onto **b**.

Find **a** the scalar projection of $3\mathbf{i} + 4\mathbf{j}$ onto $2\mathbf{i} + \mathbf{j}$,
b the vector projection of $3\mathbf{i} + 4\mathbf{j}$ onto $2\mathbf{i} + \mathbf{j}$,
c the scalar projection of $2\mathbf{i} + \mathbf{j}$ onto $3\mathbf{i} + 4\mathbf{j}$,
d the vector projection of $2\mathbf{i} + \mathbf{j}$ onto $3\mathbf{i} + 4\mathbf{j}$.

21 Find the two vectors in the $\mathbf{i} - \mathbf{j}$ plane that are perpendicular to $3\mathbf{i} - 4\mathbf{j}$ and have magnitude of 25 units.

22 Find the two unit vectors in the $\mathbf{i} - \mathbf{j}$ plane that are perpendicular to $2\mathbf{i} + \mathbf{j}$.

23 Points A, B, C and D have position vectors $2\mathbf{i} + 4\mathbf{j}$, $6\mathbf{i} + 6\mathbf{j}$, $7\mathbf{i} + 2\mathbf{j}$ and $4\mathbf{i} + \mathbf{j}$ respectively.
Find $\overrightarrow{AC}$ and $\overrightarrow{BD}$ and hence prove that AC is perpendicular to BD.

24 The position vectors of points A, B and C are $4\mathbf{i} + 7\mathbf{j}$, $6\mathbf{i} + 2\mathbf{j}$ and $8\mathbf{i} + 9\mathbf{j}$ respectively.
Find **a** $\overrightarrow{AC}$ **b** $\overrightarrow{AB}$ **c** $\overrightarrow{AC}\,.\,\overrightarrow{AB}$
d the size of angle CAB correct to the nearest degree.

25 The angle between $\mathbf{a}$ and $\mathbf{b}$ is 45°. If $\mathbf{a} = 3\mathbf{i} - \mathbf{j}$ and $\mathbf{b} = 4\mathbf{i} + y\mathbf{j}$ find the two possible values of y.

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Proofs using the scalar product

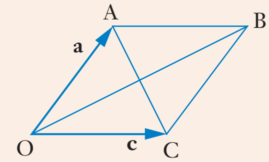
In the previous chapter, we used our understanding of vectors to prove various geometrical facts. We can now add the various properties of the scalar product to our toolbox of facts that we can use in such proofs. Useful facts include the following:

- If, for non zero vectors **a** and **b**, **a**.**b** = 0, then **a** and **b** are perpendicular vectors. (And conversely, if **a** and **b** are perpendicular vectors, **a**.**b** = 0.)
- The various properties of the scalar product listed on page 139, for example:

$$(\mathbf{a}+\mathbf{b})\,.\,(\mathbf{c}+\mathbf{d}) = \mathbf{a.c} + \mathbf{a.d} + \mathbf{b.c} + \mathbf{b.d}$$

EXAMPLE 7

To prove: The diagonals of a rhombus are perpendicular to each other.



The diagram shows a rhombus OABC. $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OC} = \mathbf{c}$.

a Find $\overrightarrow{OB}$ and $\overrightarrow{AC}$ in terms of **a** and **c**.

b With OABC a rhombus it follows that $|\mathbf{a}| = |\mathbf{c}|$.

Use this fact to prove that the diagonals of a rhombus are perpendicular to each other.

Solution

a $\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB}$
$= \mathbf{a} + \mathbf{c}$

$\overrightarrow{AC} = \overrightarrow{AO} + \overrightarrow{OC}$
$= -\mathbf{a} + \mathbf{c}$

b $\overrightarrow{OB}.\overrightarrow{AC} = (\mathbf{a}+\mathbf{c}).(-\mathbf{a}+\mathbf{c})$

$= -\mathbf{a.a} + \mathbf{a.c} - \mathbf{c.a} + \mathbf{c.c}$ Standard expansion.

$= -\mathrm{a}^2 + \mathbf{a.c} - \mathbf{a.c} + \mathrm{c}^2$ Because $\mathbf{x.x} = x^2$ and $\mathbf{x.y} = \mathbf{y.x}$.

$= \mathrm{c}^2 - \mathrm{a}^2$

$= 0$ Because, with OABC a rhombus, $|\mathbf{a}| = |\mathbf{c}|$ (i.e. a = c).

Thus $\overrightarrow{OB}$ is perpendicular to $\overrightarrow{AC}$ and hence the diagonals of a rhombus are perpendicular to each other.

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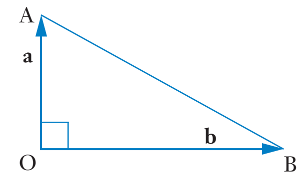
Exercise 8C

1 To prove: The theorem of Pythagoras.

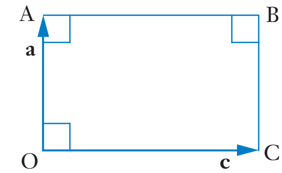


In the right triangle shown, $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$.

a Express $\overrightarrow{AB}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.

b Use the fact that $\overrightarrow{AB}.\overrightarrow{AB} = (AB)^2$ to prove that $(AB)^2 = (OA)^2 + (OB)^2$.

2 To prove: The diagonals of a rectangle are congruent.



In the rectangle shown, $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OC} = \mathbf{c}$.

a Write down the value of $\mathbf{a}.\mathbf{c}$.

b Express $\overrightarrow{AC}$ and $\overrightarrow{OB}$ in terms of $\mathbf{a}$ and $\mathbf{c}$.

c Prove that $|\overrightarrow{AC}| = |\overrightarrow{OB}|$.

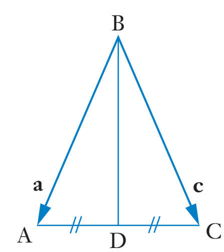
(Hint: Find $(AC)^2$ by using $\overrightarrow{AC}.\overrightarrow{AC} = (AC)^2$.)

3 To prove: If the diagonals of a parallelogram cut at right angles then the parallelogram is a rhombus.

OABC is a parallelogram with $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OC} = \mathbf{c}$.

If $\overrightarrow{AC}$ and $\overrightarrow{OB}$ are perpendicular, prove that $|\mathbf{a}| = |\mathbf{c}|$.

4 To prove: In an isosceles triangle, a line drawn from the vertex that is common to the congruent sides, to the midpoint of the third side, is perpendicular to that third side.



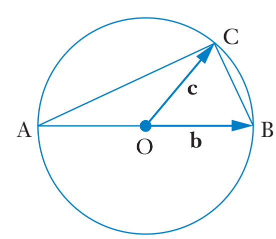
ABC is an isosceles triangle with AB = CB.

D is the midpoint of AC, $\overrightarrow{BA} = \mathbf{a}$ and $\overrightarrow{BC} = \mathbf{c}$.

a Find $\overrightarrow{AC}$ and $\overrightarrow{BD}$ in terms of $\mathbf{a}$ and $\mathbf{c}$.

b Use $\overrightarrow{AC}.\overrightarrow{BD}$ to prove that $\angle BDA$ is a right angle.

5 To prove: The angle in a semicircle is a right angle.



The diagram shows a circle centre O with AB a diameter and C a point on the circumference.

$\overrightarrow{OB} = \mathbf{b}$ and $\overrightarrow{OC} = \mathbf{c}$.

a Express $\overrightarrow{CB}$, $\overrightarrow{AO}$ and $\overrightarrow{AC}$ in terms of $\mathbf{b}$ and/or $\mathbf{c}$.

b Prove that $\angle ACB$ is a right angle.

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6 Prove that the sum of the squares of the lengths of the diagonals of a parallelogram is equal to the sum of the squares of the lengths of the sides.

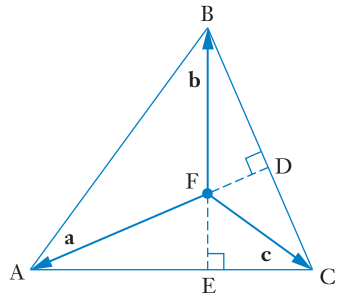
7 To prove: The altitudes of a triangle are concurrent.

The diagram shows triangle ABC.

The perpendicular from A to BC meets BC at D and the perpendicular from B to AC meets AC at E.

AD and BE intersect at F.

Use the facts that $\mathbf{a}.\overrightarrow{BC} = 0$ and $\mathbf{b}.\overrightarrow{AC} = 0$ to prove that $\overrightarrow{CF}$ is perpendicular to $\overrightarrow{AB}$.



8 To prove: The diagonals of a rhombus bisect the angles of the rhombus.

The diagram shows a rhombus OABC with $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OC} = \mathbf{c}$.

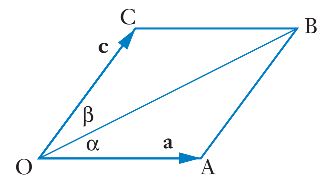
$\angle BOA = \alpha$ and $\angle BOC = \beta$.

a Writing $|\mathbf{a}|$ as a and $|\mathbf{c}|$ as c, prove that $\overrightarrow{OB}.\overrightarrow{OA} = a^2 + \mathbf{a}.\mathbf{c}$

and $\overrightarrow{OB}.\overrightarrow{OC} = c^2 + \mathbf{a}.\mathbf{c}$

b Hence prove that $\alpha = \beta$.

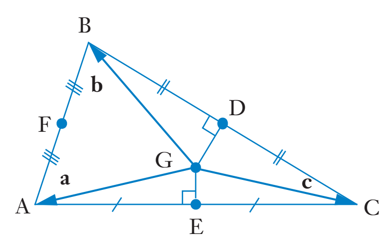
(Note: The above proves that BO bisects $\angle COA$. From the properties of isosceles triangles it follows that BO also bisects $\angle CBA$. By repeating this process at C the initial statement is proved.)



9 To prove: The perpendicular bisectors of the sides of a triangle are concurrent.

The diagram shows triangle ABC. Points D, E and F are the midpoints of BC, AC and AB respectively. The perpendicular bisector of BC meets the perpendicular bisector of AC at G.

Vectors **a**, **b** and **c** are as shown in the diagram, and $|\mathbf{a}| = a$, $|\mathbf{b}| = b$ and $|\mathbf{c}| = c$.



a Use the fact that $\overrightarrow{GE}.\overrightarrow{AC} = 0$ to prove $a^2 = c^2$.

b Use the fact that $\overrightarrow{GD}.\overrightarrow{BC} = 0$ to prove $b^2 = c^2$.

c By determining $\overrightarrow{GF}.\overrightarrow{AB}$, prove that GF is perpendicular to AB.

Miscellaneous exercise eight

This miscellaneous exercise may include questions involving the work of this chapter, the work of any previous chapters, and the ideas mentioned in the Preliminary work section at the beginning of the book.

1 Discuss the validity of each of the following statements.

- Quadrilateral ABCD is a rhombus $\Leftrightarrow$ Quadrilateral ABCD is a parallelogram.
- Diagonals of quadrilateral PQRS cut at right angles $\Leftrightarrow$ PQRS is a rhombus.
- The diagonals of parallelogram WXYZ cut at right angles $\Leftrightarrow$ WXYZ is a rhombus.

2 If $\mathbf{a}$ is perpendicular to $(\mathbf{b} - \mathbf{a})$, which of the following statements are necessarily true?

a $\mathbf{a}.(\mathbf{b} - \mathbf{a}) = 0$ **b** $\mathbf{a}.\mathbf{b} = \mathbf{a}.\mathbf{a}$ **c** $\mathbf{a} = \mathbf{b}$ **d** $\mathbf{a}.\mathbf{b} = |\mathbf{a}|^2$

3 Points A and B have position vectors $\mathbf{i} + 3\mathbf{j}$ and $21\mathbf{i} - 7\mathbf{j}$ respectively.

Find the position vector of the point P such that $\overrightarrow{AP} : \overrightarrow{PB} = 3:7$.

4 How many different groups of eight letters are there if each group must contain two vowels, six consonants and no letter used more than once?

5 Point A has position vector $-4\mathbf{i} + 6\mathbf{j}$.

Relative to point A a second point, B, has location $6\mathbf{i} - \mathbf{j}$.

Relative to point B a third point, C, has location $4\mathbf{i} + 5\mathbf{j}$.

a How far is point C from point A? (Give your answer in exact form.)

b What is the position vector of point C?

c Find the position vector of the midpoint of the line AC.

6 Three digit numbers are to be made using the digits 1, 2, 3, 4, 5.

How many three digit numbers are possible if

a each digit can be used more than once in a number?

b each digit may not be used more than once in a number?

c multiple use of a digit is not allowed and the number must be even?

d multiple use of a digit is not allowed and the number must be odd?

e multiple use of a digit is not allowed and the number must be odd and more than 300?

7 Given that $|\mathbf{c}| = 2$, $|\mathbf{d}| = 3$ and $\mathbf{c}.\mathbf{d} = -5$, find:

a the angle between $\mathbf{c}$ and $\mathbf{d}$, correct to the nearest degree,

b $\mathbf{c}.\mathbf{c}$

c $\mathbf{d}.\mathbf{d}$

d $(\mathbf{c} + \mathbf{d}).(\mathbf{c} + \mathbf{d})$

e $|\mathbf{c} + \mathbf{d}|$

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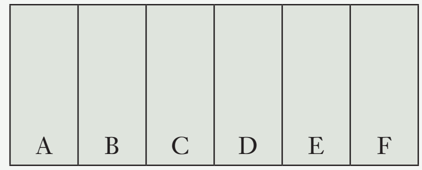
8 Six files, A, B, C, D, E and F are to be arranged on a shelf.



a In how many ways can this be done?

In how many of the arrangements

b is file A at the extreme left?

c is file A next to file B?

d are the first three files on the left, A, B and C in that order?

e are the first three files on the left, A, B and C in any order?

f are files A, B, C and D together in that order?

g are files A, B, C and D together in any order?

9 One section of a river runs from North West to South East with speed 1 m/s.

A person wishes to row a boat from a point A on one bank to a point B on the other bank, B being due North of A.

The person can row the boat with a speed of 2 m/s in still water and the river has constant width of 30 metres.

a On what bearing should the person row the boat so that their effort, together with the flow of the river, produces the desired result (answer to the nearest degree)?

b How long would the journey take (to the nearest second)?

10 In how many ways can Amanda, Bridie, Claire, Donelle, Erin, Fran, Gia, Harni and Icolyn be arranged in a row for a photograph if:

a Donelle is to be in the middle?

b Donelle is to be in the middle and Erin and Harni are each to be at an end?

c Donelle is to be in the middle, Erin and Harni must each be at an end and Claire and Icolyn are to stand next to each other?

11 Points A, B and C have position vectors $2\mathbf{p} + \mathbf{q}$, $3\mathbf{p} - \mathbf{q}$ and $6\mathbf{p} - 7\mathbf{q}$.

Prove that A, B and C are collinear and find the ratios $\overrightarrow{AB} : \overrightarrow{BC}$ and $\overrightarrow{AB} : \overrightarrow{AC}$.

12 A committee consists of 13 people: 6 men and 7 women. Five of these thirteen are to be randomly selected to form a sub-committee, and then these chosen five are to be arranged in a line for a photograph.

a How many different photographs are possible?

b How many of these consist of two men and three women?

13 The position vectors of points D, E and F are $-3\mathbf{i} + 3\mathbf{j}$, $3\mathbf{i} + 2\mathbf{j}$ and $8\mathbf{i} + 7\mathbf{j}$ respectively.

Find **a** $\overrightarrow{ED}$ **b** $\overrightarrow{EF}$ **c** $\overrightarrow{ED} \cdot \overrightarrow{EF}$

d the size of angle DEF correct to the nearest degree.

14 The angle between $\mathbf{a}$ $(= 5\sqrt{3}\mathbf{i} + \mathbf{j})$ and $\mathbf{b}$ $(= 2\sqrt{3}\mathbf{i} + w\mathbf{j})$ is 60°.

With the assistance of a calculator, find the two possible values of w.

15 Vectors **a**, **b** and **c** are such that $\mathbf{a} = 2\mathbf{i} - 3\mathbf{j}$, $\mathbf{b} = x\mathbf{i} + 4\mathbf{j}$ and $\mathbf{c} = 9\mathbf{i} - y\mathbf{j}$.
If **a** and **b** are perpendicular and **a** and **c** are parallel, determine x and y.

16 Find the resultant of forces **F** and **P** where $\mathbf{F} = (6\mathbf{i} + 4\mathbf{j})$ N
and $\mathbf{P} = (2\mathbf{i} - 7\mathbf{j})$ N.
Determine the angle between this resultant and whichever of **F** and **P** has the smaller magnitude, giving your answer to the nearest 0.1°.

17 In the parallelogram OABC, $\overrightarrow{OA} = \mathbf{a}$, and $\overrightarrow{OC} = \mathbf{c}$. D is the midpoint of CB and E is a point on OC such that $\overrightarrow{OE} = \frac{1}{3}\overrightarrow{OC}$. The lines OD and AE intersect at M. If $\overrightarrow{AM} = h\overrightarrow{AE}$ and $\overrightarrow{OM} = k\overrightarrow{OD}$ determine h and k.

18 **a** **i** How many different six letter 'words' can be formed using letters chosen from the word HARLEQUIN with no letter being used more than once in a word?

ii How many of these six letter words contain at least one vowel?

b **i** How many different six letter 'words' can be formed using letters chosen from the word PORTHCAWL with no letter being used more than once in a word?

ii How many of these six letter words contain at least one vowel?

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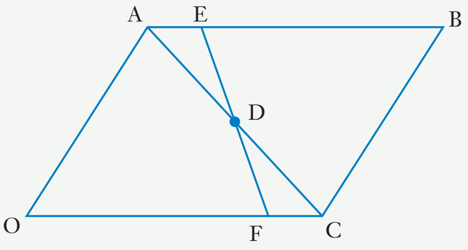
19 In how many ways can twelve people be sorted into two groups of six?

(Note: The two groups are not numbered or labelled in any way. Thus the two groups {A, B, C, D, E, F} and {G, H, I, J, K, L} are not considered different to {G, H, I, J, K, L} and {A, B, C, D, E, F}.)

20 In parallelogram OABC, $\overrightarrow{OA} = \mathbf{a}$, and $\overrightarrow{OC} = \mathbf{c}$.
Point D is the mid-point of AC and E is a point on AB such that $\overrightarrow{AE} = h\overrightarrow{AB}$.
The line drawn from E, through D, meets OC at F with $\overrightarrow{CF} = k\overrightarrow{CO}$.



a Obtain an expression for $\overrightarrow{ED}$ in terms of **a**, **c** and h.

b Obtain an expression for $\overrightarrow{DF}$ in terms of **a**, **c** and k.

c If $\overrightarrow{ED} = m\overrightarrow{DF}$ prove that

i $m = 1$ (i.e. $\overrightarrow{ED} = \overrightarrow{DF}$),

ii $h = k$.

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